Squeezing on momentum states for atom interferometry: Supplemental Material

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I. PHOTON SHOT NOISE LIMITED ATOM NUMBER RESOLUTION

In this Section, the details of the derivation that lead to the expression of the phase shift $\Delta\phi_{\mathrm{ph}}$ in Equation (2) of the main text are provided.

In order to obtain this result, a simpler system is first considered, with a single transition from a ground to an optically excited state. The result will then be generalized to the scheme described in the main text, with four atomic levels, two momentum states in the ground ${}^1S_0$ level and two momentum states for the excited ${}^3P_1$ level (see Fig. 1 b) of the main text).

The simplified system can be represented as a collection of N two-level atoms each with ground state $|g\rangle_i$ and optically excited state $|e\rangle_i$, where the index $i = 1, \cdots, N$ labels the atoms. The energy of the transition $|g\rangle_i - |e\rangle_i$ between the two atomic levels is denoted by $\hbar\omega_0$ and the excited state decay rate is denoted by Γ. The atomic ensemble is probed through the light field, at frequency ω_r, circulating in an optical cavity with resonance frequency ω_c and photon decay rate κ. The detuning of the probe laser from atomic resonance will be indicated as $\Delta = \omega_r - \omega_0$ and the detuning of the probe from cavity resonance will be indicated as $\delta = \omega_r - \omega_c$.

The evolution of the coupled atoms-cavity system can be described in the framework of input-output theory [1, 2] through the equations of motion for the variables $\sigma_- = \langle \sum_{i=1}^N |g\rangle_i \langle e|_i \rangle$ and c, the expectation values of the collective (atomic) ladder operator and

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of the annihilation operator of the cavity field, respectively. In a frame rotating at the frequency of the probe laser ω_r, these are written as

$$\frac{d}{dt}\sigma_- = i\left(\Delta + i\frac{\Gamma}{2}\right)\sigma_- - igNc, \tag{1}$$

$$\frac{d}{dt}c = i\left(\delta + i\frac{\kappa}{2}\right)c - ig\sigma_- - \sqrt{\kappa_{\text{in}}}\beta, \tag{2}$$

where β is the amplitude of the field that is incident onto the optical cavity, with units of $\sqrt{\text{photons/s}}$. With reference to Fig. 1 a) of the main text, the input field is such that $\langle \hat{b}_{\text{in}} \rangle = \beta$. Moreover, κ_{in} is the contribution of the input mirror transmission to the total photon loss rate κ. These equations are written assuming negligible excitation of the two-level systems.

The steady-state response of the system is then derived for vanishing time derivatives: $\frac{d}{dt}\sigma_- = 0$, $\frac{d}{dt}c = 0$. In particular, the cavity field is expressed by

$$c = -i\frac{2\frac{\sqrt{\kappa_{\text{in}}}}{\kappa}\beta}{\frac{2\delta}{\kappa} + N\eta\mathcal{L}_d(\Delta) + i[1 + N\eta\mathcal{L}_a(\Delta)]}, \tag{3}$$

where $\mathcal{L}_a(\Delta) = \Gamma^2/(\Gamma^2 + 4\Delta^2)$ and $\mathcal{L}_d(\Delta) = -2\Delta\Gamma/(\Gamma^2 + 4\Delta^2)$ are the absorption and dispersion profiles, respectively, and $\eta = 4g^2/(\kappa\Gamma)$ is the single-atom cooperativity.

The collective measurement of the atomic state populations is performed, as seen in Fig. 1 a) of the main text, by detecting the light, with amplitude $\langle \hat{b}_{\text{out}} \rangle$, that is reflected from the optical cavity. This is the superposition of the amplitude of the light reflected from the input mirror $\sqrt{R_{\text{in}}}\beta$ and of the light that entered the cavity and is transmitted after a round trip from the input mirror $\sqrt{\kappa_{\text{in}}}c$. For a high-finesse cavity, the reflection coefficient can be approximated by $R_{\text{in}} \simeq 1$. As a result, the output field is expressed as

$$\langle \hat{b}_{\text{out}} \rangle = \beta - i\frac{2\frac{\kappa_{\text{in}}}{\kappa}\beta}{\frac{2\delta}{\kappa} + N\eta\mathcal{L}_d(\Delta) + i[1 + N\eta\mathcal{L}_a(\Delta)]}. \tag{4}$$

As anticipated, these results can be extended to the scheme considered in Fig. 1 b) of the main text. The two optical transitions $|^1S_0, 0\rangle - |^3P_1, \hbar k_r\rangle$ and $|^1S_0, 2n\hbar k_b\rangle - |^3P_1, 2n\hbar k_b + \hbar k_r\rangle$, with frequency splitting $2\delta\omega_r$, are considered and an effective spin-1/2 is associated with the two momentum components of the ground state with spin down $|\downarrow\rangle \equiv |^1S_0, 0\rangle$ and spin up $|\uparrow\rangle \equiv |^1S_0, 2n\hbar k_b\rangle$. The collective spin is given by $S = N/2$ and the population difference between the number $N_\uparrow$ of atoms in state $|\uparrow\rangle$ and the number $N_\downarrow$ of atoms in state $|\downarrow\rangle$ is given by $N_\uparrow - N_\downarrow = 2S_z$.

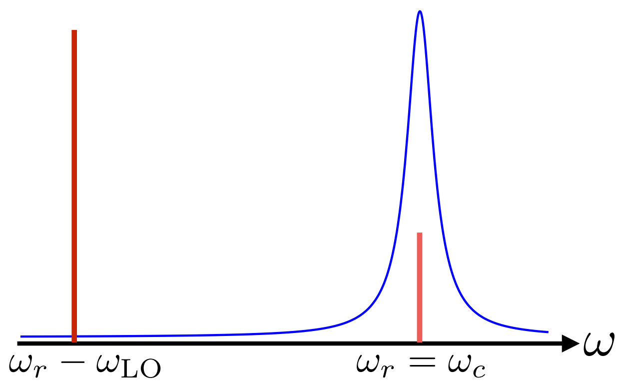

FIG. 1. Conceptual scheme for the measurement of the photon phase shift of the light reflected from the cavity. Blue line: intracavity photon number for the empty cavity, red segments: incident laser spectrum. The phase shift $\Delta\phi_{\rm ph}$ is detected through the interference of the probe light at frequency ω_r, resonant with the bare cavity mode at frequency ω_c, with a strong field at frequency $\omega_r - \omega_{\rm LO}$. (see text for more details)

With this notation, the expression for the output field in Eq. (4) can be written for the case where the probe laser is tuned halfway between the two optical transitions. In the limit of negligible atomic excitation, the polarizabilities of the optical transitions are additive [3] and the following replacements can be performed:

$$N\mathcal{L}_a(\Delta) \to (S - S_z)\,\mathcal{L}_a(\delta\omega_r) + (S + S_z)\,\mathcal{L}_a(-\delta\omega_r) = 2S\mathcal{L}_a(\delta\omega_r) \tag{5}$$

$$N\mathcal{L}_d(\Delta) \to (S - S_z)\,\mathcal{L}_d(\delta\omega_r) + (S + S_z)\,\mathcal{L}_d(-\delta\omega_r) = -2S_z\mathcal{L}_d(\delta\omega_r). \tag{6}$$

These equalities hold because $\mathcal{L}_a$ and $\mathcal{L}_d$ are even and odd functions, respectively.

The population difference can be measured through the photon phase shift of the light reflected from the cavity. In Fig. 1, a possible scheme for the phase shift measurement is shown that is conceptually similar to the Pound-Drever-Hall method. This relies on probing the cavity resonance, with $\omega_r = \omega_c$, and on detecting the interference of the reflected light with a strong frequency component that is offset from cavity resonance by the local oscillator frequency $\omega_{\rm LO}$. By denoting the amplitude of the strong local oscillator as $\beta_{\rm LO}e^{-i[(\omega_r-\omega_{\rm LO})t+\phi_{\rm LO}]}$ and by considering the interference with the field reflected from the cavity $\langle\hat{b}_{\rm out}\rangle\, e^{-i\omega_r t}$, the photon flux of detected photons $\beta^2_{\rm det}$ (in units of photons/s) can be written as

$$\beta^2_{\rm det} = \underbrace{\langle \hat{b}_{\rm out}\rangle^2 + \beta^2_{\rm LO}}_{\equiv \mathcal{N}^2/T_m} + \underbrace{\left\{ \beta\beta_{\rm LO} e^{i(\omega_{\rm LO}t+\phi_{\rm LO})} \left[1 - \frac{2i\frac{\kappa_{\rm in}}{\kappa}}{i[1+N\eta\mathcal{L}_a(\delta\omega_r)] - 2S_z\eta\mathcal{L}_d(\delta\omega_r)} \right] + c.c. \right\}}_{\equiv \mathcal{S}/T_m}. \tag{7}$$

The two terms in curly brackets account for the interference between the two fields, that is identified as the *signal* $\mathcal{S}$ and can be written as

$$\frac{\mathcal{S}}{T_m} = 2\beta\beta_{\rm LO} \frac{2\frac{\kappa_{\rm in}}{\kappa} - 1 - N\eta\mathcal{L}_a(\delta\omega_r)}{1+N\eta\mathcal{L}_a(\delta\omega_r)} \cos(\omega_{\rm LO}t + \phi_{\rm LO} + \pi - \Delta\phi_{\rm ph}). \tag{8}$$

In this form, Eq. (8) suggests that $\mathcal{S}$ is the number of signal photons collected during the measurement time T_m.

The photon phase shift $\Delta\phi_{\rm ph}$ is determined by the argument of $\mathcal{S}$, is given by

$$\Delta\phi_{\rm ph} = 4\frac{\kappa_{\rm in}}{\kappa} \frac{S_z\eta\mathcal{L}_d(\delta\omega_r)}{[2\frac{\kappa_{\rm in}}{\kappa} - 1 - N\eta\mathcal{L}_a(\delta\omega_r)][1+N\eta\mathcal{L}_a(\delta\omega_r)]}, \tag{9}$$

which is the expression in Eq. (2) of the main text, showing the relation between the momentum state population difference $2S_z$ and the photon phase shift $\Delta\phi_{\rm ph}$. We note that this expression is valid for $\delta\omega_r \gg |S_z|\Gamma$. For an initial coherent state that is an equal superposition of the two momentum states, an upper limit on S_z can be estimated by the atomic shot noise fluctuations, so in other terms, Eq. (9) is valid for $\delta\omega_r \gg \Gamma\sqrt{N}/2$.

The photon phase shift is measured by mixing the detected electronic signal with the local oscillator source and by tuning the phase $\phi_{\rm LO}$ in order to maximize the sensitivity of $\mathcal{S}$ to variations of $\Delta\phi_{\rm ph}$.

The next step is the determination of the photon shot noise limited atom number resolution $2(\Delta S_z)^2/S$ in Eq. (3) of the main text. The photon number variance $\Delta\mathcal{N}^2$ is determined by recalling that, for an initial equal superposition of the states $|\uparrow\rangle$ and $|\downarrow\rangle$, the average spin component $\langle S_z\rangle = 0$. The photon number variance is therefore given by $\Delta\mathcal{N}^2 = (\langle\hat{b}_{\rm out}\rangle^2 + \beta^2_{\rm LO})T_m \simeq \beta^2_{\rm LO}T_m$ (see Eq. (7)). If the photodetector operates at the shot noise limit, the atom number resolution is determined by the condition $(\mathcal{S}/\Delta\mathcal{N})^2 = 1$.

In order to derive the squeezing limits from the collective population measurement, it is convenient to express the number of incident photons in the probe field $n_{\rm in} = \beta^2 T_m$ in terms of the number of photons scattered into free space per atom $n_{\rm sc}$. This conversion is achieved by first considering the atomic scattering rate $\Gamma_{\rm sc} = \Gamma s\mathcal{L}_a(\delta\omega_r)/2$, where $s = 2\Omega^2/\Gamma^2$ is the saturation parameter and $\Omega = 2g|c|$ is the Rabi frequency [4]. With $n_{\rm sc} = \Gamma_{\rm sc} T_m$ and by inserting the expression of the cavity field amplitude Eq. (3),

$$\frac{n_{\rm sc}}{n_{\rm in}} = \frac{4\eta\frac{\kappa_{\rm in}}{\kappa}\mathcal{L}_a(\delta\omega_r)}{[1+N\eta\mathcal{L}_a(\delta\omega_r)]^2}. \tag{10}$$

The squared atom number resolution $(2\Delta S_z)^2$, normalized to the atom shot noise variance $2S$ is then given by

$$\frac{2(\Delta S_z)^2}{S} = \frac{[1+N\eta\mathcal{L}_a(\delta\omega_r)]^2\mathcal{L}_a(\delta\omega_r)}{4N\eta\frac{\kappa_{\rm in}}{\kappa}n_{\rm sc}[\mathcal{L}_d(\delta\omega_r)]^2}. \tag{11}$$

The term $\kappa_{\rm in}/\kappa$ is the fraction of detected-to-incident photons when the cavity is empty and $\omega_r = \omega_c$. This term can therefore be interpreted as the detection efficiency ϵ_d. In the definition of the detection efficiency, the effect of cavity losses, of detector quantum efficiency and of any additional loss in the path from the cavity input mirror to the photodetector are included. With $\kappa_{\rm in}/\kappa \to \epsilon_d$, the atom number resolution in Eq. (3) of the main text is derived.

II. LIMITS ON ATTAINABLE SQUEEZING

The optimum squeezing attainable through the collective population measurement, expressed by Eq. (4) in the main text, is set by scattering into free space. For the considered scheme, after scattering one photon into free space, the coherent momentum state superposition of an atom is destroyed and the resulting recoil in a random direction will in general cause the trajectory to deviate from the vertical direction. As a result, the atom is lost from the interferometer even though it will be detected in the nondestructive pre-measurement. Photon scattering into free space additionally causes diffusion of the spin component S_z.

The amount of added spin noise is determined by computing the population imbalance arising from the scattering of $p_{\rm sc} = Nn_{\rm sc}$ photons into free space, in the limit $p_{\rm sc} \ll N$. Following the binomial distribution, the probability for scattering l photons from atoms in $|^1S_0, 0\rangle$ and $p_{\rm sc} - l$ photons from atoms in $|^1S_0, 2n\hbar k_b\rangle$ is given by

$$P_l = \frac{1}{2^{p_{\rm sc}}}\begin{pmatrix} p_{\rm sc} \\ l \end{pmatrix}. \tag{12}$$

In this expression, the probability for scattering from one of the two momentum states is set to 1/2. The mean and variance of the distribution are given by $\langle l\rangle = p_{\rm sc}/2$ and $\mathrm{Var}(l) =$

$p_{\rm sc}/4$, respectively. As a result, the spin variance increase due to free space scattering can be written as

$$(2\Delta S_z)^2_{\rm sc} = {\rm Var}(p_{\rm sc} - 2l) = p_{\rm sc}. \tag{13}$$

The squeezing limit can then be determined by summing the atom number resolution of the measurement given by Eq. (11) and the relative variance increase $2(\Delta S_z)^2_{\rm sc}/S$. This gives the total atom number variance normalized to the atom shot noise as

$$\left[\frac{2(\Delta S_z)^2}{S}\right]_{\rm tot} = \frac{[1 + N\eta\mathcal{L}_a(\delta\omega_r)]^2\mathcal{L}_a(\delta\omega_r)}{4N\eta\epsilon_d n_{\rm sc}[\mathcal{L}_d(\delta\omega_r)]^2} + n_{\rm sc}. \tag{14}$$

The above expression is minimized by the optimum value of $n_{\rm sc}$ given by Eq. (4) in the main text:

$$n_{\rm sc} = \sqrt{\frac{[1 + N\eta\mathcal{L}_a(\delta\omega_r)]^2\mathcal{L}_a(\delta\omega_r)}{4N\eta\epsilon_d[\mathcal{L}_d(\delta\omega_r)]^2}}. \tag{15}$$

The resulting optimum squeezing is given by

$$\left[\frac{2(\Delta S_z)^2}{S}\right]_{\rm opt} = \sqrt{\frac{[1 + N\eta\mathcal{L}_a(\delta\omega_r)]^2\mathcal{L}_a(\delta\omega_r)}{N\eta\epsilon_d[\mathcal{L}_d(\delta\omega_r)]^2}}. \tag{16}$$

In the dispersive limit $N\eta\mathcal{L}_a(\delta\omega_r) \ll 1$, the metrological gain reaches the maximum value $\xi_m = \sqrt{N\eta\epsilon_d}$.

In deriving the metrological gain, the loss of contrast $\mathcal{C}$ due to scattering into free space should also be considered. However, as a function of $n_{\rm sc}$, the contrast decays as $\mathcal{C} = e^{-n_{\rm sc}}$ [2] and, for realistic values of the parameters, the optimum $n_{\rm sc} \ll 1$. In this limit, the effect of contrast loss can be neglected.

III. SQUEEZING ENHANCEMENT BY ELECTROMAGNETICALLY INDUCED TRANSPARENCY

In this Section, squeezing enhancement by electromagnetically induced transparency (EIT) is considered by deriving the modified atomic response. As explained in the main text, EIT can be generated through Raman coupling of the $^3P_1$ and $^3P_0$ states via the $^3S_1$ state. A possible practical scheme, closely related to the diagram in Fig. 3 a) of the main text, is formed by a probe light with linear polarization that couples to the $|^3P_1, m_J = 0\rangle$ state and by two Raman beams R1 and R2 with Rabi frequencies $\Omega_{\rm R1}$ and $\Omega_{\rm R2}$, respectively. The field R1, with frequency ω_1, couples the states $|^3P_0\rangle$ and $|^3S_1, m_J = +1\rangle$, whereas the field R2, with frequency ω_2, couples the states $|^3P_1, m_J = 0\rangle$ and $|^3S_1, m_J = +1\rangle$. It is notationally convenient to make the replacements $|^1S_0\rangle \to |a\rangle$, $|^3P_1, m_J = 0\rangle \to |b\rangle$, $|^3P_0\rangle \to |c\rangle$, $|^3S_1, m_J = +1\rangle \to |d\rangle$ and write the atomic Hamiltonian as

$$\hat{\mathcal{H}} = \hbar\omega_a |a\rangle\langle a| + \hbar\omega_b |b\rangle\langle b| + \hbar\omega_c |c\rangle\langle c| + \hbar\omega_d |d\rangle\langle d| + \frac{\hbar\Omega}{2}(|b\rangle\langle a| e^{-i\omega_r t} + |a\rangle\langle b| e^{i\omega_r t})$$
$$+\frac{\hbar\Omega_{\rm R1}}{2}(|d\rangle\langle c| e^{-i\omega_1 t} + |c\rangle\langle d| e^{i\omega_1 t}) + \frac{\hbar\Omega_{\rm R2}}{2}(|d\rangle\langle b| e^{-i\omega_2 t} + |b\rangle\langle d| e^{i\omega_2 t}), \tag{17}$$

where $\hbar\omega_i$ is the unperturbed atomic energy of state $|i\rangle$ and Ω is the probe Rabi frequency.

Using first-order perturbation theory for the weak probe field, $\Omega \ll \Gamma$, a closed set of equations of motion for the matrix elements $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$ of the density operator $\hat{\rho}$ is written as

$$\dot{\rho}_{ab} = i\omega_{ba}\rho_{ab} - \frac{\Gamma}{2}\rho_{ab} + i\frac{\Omega_{\rm R2}}{2}\rho_{ad}e^{-i\omega_2 t} + i\frac{\Omega}{2}e^{i\omega_r t} \tag{18}$$

$$\dot{\rho}_{ac} = i\omega_{ca}\rho_{ac} + i\frac{\Omega_{\rm R1}}{2}\rho_{ad}e^{-i\omega_1 t} \tag{19}$$

$$\dot{\rho}_{ad} = i\omega_{da}\rho_{ad} + i\frac{\Omega_{\rm R1}}{2}\rho_{ac}e^{i\omega_1 t} + i\frac{\Omega_{\rm R2}}{2}\rho_{ab}e^{i\omega_2 t}, \tag{20}$$

where $\omega_{ij} = \omega_i - \omega_j$ is the frequency of the transition $|j\rangle \to |i\rangle$ [5].

Two-photon Raman coupling is favored in comparison to single-photon transitions when the detuning of the Raman fields from resonance is large compared to the excited $^3S_1$ state decay rate and compared to the single-photon Rabi frequencies $\Omega_{\rm R1}$ and $\Omega_{\rm R2}$. In this regime, it is possible to adiabatically eliminate the excited $^3S_1$ state. This is achieved by defining in (20) the new variables $\sigma_{ab} = \rho_{ab}e^{-i\omega_{ba}t}$, $\sigma_{ac} = \rho_{ac}e^{-i\omega_{ca}t}$, $\sigma_{ad} = \rho_{ad}e^{-i\omega_{da}t}$ and by observing that σ_{ab} and σ_{ac} are slowly varying compared to σ_{ad}. The resulting equation can be directly integrated:

$$\sigma_{ad} = \frac{\Omega_{\rm R1}}{2\Delta_{\rm R}}e^{i\Delta_1 t}\sigma_{ac} + \frac{\Omega_{\rm R2}}{2\Delta_{\rm R}}e^{i\Delta_2 t}\sigma_{ab}, \tag{21}$$

where $\Delta_1 = \omega_1 - \omega_{dc}$, $\Delta_2 = \omega_2 - \omega_{db}$ and, in the denominators, the approximation $\Delta_1 \simeq \Delta_2 \equiv \Delta_{\rm R}$ is made, thus neglecting the difference between the two detunings. Substitution into equations (18) and (19) yields the effective three-level equations of motion

$$\dot{\rho}_{ab} = i\omega'_{ba}\rho_{ab} - \frac{\Gamma}{2}\rho_{ab} + i\frac{\Omega}{2}e^{i\omega_r t} + i\frac{\Omega_{\rm eff}}{2}\rho_{ac}e^{i\omega_{\rm R} t} \tag{22}$$

$$\dot{\rho}_{ac} = i\omega'_{ca}\rho_{ac} + i\frac{\Omega_{\rm eff}}{2}\rho_{ab}e^{-i\omega_{\rm R} t}. \tag{23}$$

Here $\Omega_{\rm eff} = \Omega_{\rm R1}\Omega_{\rm R2}/(2\Delta_{\rm R})$ is the effective two-photon Rabi frequency, $\omega_{\rm R} = \omega_1 - \omega_2$ is the frequency difference between the two Raman lasers and $\omega'_{ba} = \omega_{ba} + \Omega_{\rm R2}^2/(4\Delta_{\rm R})$, $\omega'_{ca} =$

$\omega_{ca}+\Omega_{\rm R1}^2/(4\Delta_{\rm R})$ are the transition frequencies corrected for the AC Stark shift of the Raman fields.

The effect of electromagnetically induced transparency is seen by solving (22) and (23) for the steady state solution. This is achieved by defining $\rho_{ab} = \tilde{\sigma}_{ab} e^{i\omega_r t}$ and $\rho_{ac} = \tilde{\sigma}_{ac} e^{i(\omega_r - \omega_{\rm R})t}$ and by setting $\dot{\tilde{\sigma}}_{ab} = \dot{\tilde{\sigma}}_{ac} = 0$. The relevant coherence between the states $|a\rangle$ and $|b\rangle$ is then found to be

$$\rho_{ab} = i \frac{\Omega/2}{\frac{\Gamma}{2} + i\left(\Delta - \frac{\Omega_{\rm eff}^2}{4(\Delta-\delta')}\right)} e^{i\omega_r t}. \tag{24}$$

The detuning $\Delta = \omega_r - [\omega_{ba} + \Omega_{\rm R2}^2/(4\Delta_{\rm R})]$ in this case also accounts for the AC Stark shift of the field R2 and δ' is the two-photon Raman detuning given by

$$\delta' = \omega_{\rm R} - \left(\omega_{bc} + \frac{\Omega_{\rm R2}^2}{4\Delta_{\rm R}} - \frac{\Omega_{\rm R1}^2}{4\Delta_{\rm R}}\right). \tag{25}$$

In the absence of EIT, Eq. (24) with $\Omega_{\rm eff} = 0$ describes the atomic absorption and dispersion features already considered in the previous sections. The effect of induced transparency is seen from the additional term $\Omega_{\rm eff}^2/(4\Delta)$ that is subtracted, for $\delta' = 0$, from the single-photon detuning Δ. As a result, the dispersive $\mathcal{L}_d(\Delta)$ and absorption $\mathcal{L}_a(\Delta)$ profiles are described, in the presence of EIT, by computing these functions at the effective detuning $\Delta_E = \Delta - \Omega_{\rm eff}^2/(4\Delta)$.

The validity of adiabatic elimination in this system was verified numerically by accounting for the population and decay of the excited $^3S_1$ state. In this computation, the losses in the metastable $^3P_2$ state were considered. The decay and losses were treated by the Monte-Carlo wavefunction method [6]. The validity of adiabatic elimination for this scheme was confirmed by observing quantitative agreement between the computation results and the effective three-level model derived above.

The replacement $\Delta \to \Delta_E$ in Eq. (4) yields the expression of the output field $\langle \hat{b}_{\rm out}\rangle$ in the presence of EIT. By then repeating the same procedure that led to Eq. (11) and Eq. (16), it is seen that, in generalizing the results to the optical transitions involving the two momentum states, the Doppler splitting $\delta\omega_r$ should be replaced with $\delta\omega_E = \delta\omega_r - \Omega_{\rm eff}^2/(4\delta\omega_r)$, when the two-photon resonance $\delta' = 0$ is fulfilled, as stated in the main text.

IV. PHOTON SCATTERING FROM THE EXCITED $^3S_1$ STATE IN RAMAN-DRIVEN ELECTROMAGNETICALLY INDUCED TRANSPARENCY

In this Section, a criterion for the choice of the Raman laser intensity and detuning that avoids additional scattering into free space (with a consequent squeezing reduction) is provided. The average $^3S_1$ population in the limit $\Omega_{\rm R1}, \Omega_{\rm R2} \ll |\Delta_{\rm R}|$ is given by [4]

$$P(^3S_1) = \frac{\Omega_{\rm R1}^2 + \Omega_{\rm R2}^2}{4\Delta_{\rm R}^2} P_{\rm exc}, \tag{26}$$

where $P_{\rm exc}$ is the average population of the $^3P_1$ state. This expression is written by accounting for the fact that, in the presence of Raman coupling, $P_{\rm exc}$ is also the average population of the $^3P_0$ state. With the assumption that $\Omega_{\rm R1} = \Omega_{\rm R2} \equiv \Omega_{\rm R}$ and that $n_{\rm sc}^L$ is the upper limit for the number of photons per atom scattered into free space per atom by decay from the $^3S_1$ state, the corresponding limit on the ratio $\Omega_{\rm R}/\Delta_{\rm R}$ between the single-photon Rabi frequency and the Raman detuning is then estimated as

$$\left(\frac{\Omega_{\rm R}}{\Delta_{\rm R}}\right)_L = \sqrt{\frac{2n_{\rm sc}^L}{\Gamma_T P_{\rm exc} T_m}}. \tag{27}$$

For the $^3S_1$ state, $\Gamma_T \simeq 2\pi \times 12.4$ MHz is the total decay rate. The parameters of the main text are considered, with a measurement time duration $T_m = 200$ μs, $P_{\rm exc} = 5 \times 10^{-4}$ and $n_{\rm sc}^L = 5\times 10^{-5}$ which is a hundred times smaller than the value of $n_{\rm sc}$ that yields the optimum squeezing. As a result, $(\Omega_{\rm R}/\Delta_{\rm R})_L \simeq 4\times 10^{-3}$. The requirements on Raman detuning for the proposed scheme are less severe than for standard Raman transitions because of the reduced population of the $^3P_1$ state. This small population yields a reduction in the scattering rate by a factor $P_{\rm exc}$. This improvement in turn allows to operate the Raman lasers closer to atomic resonance with a resulting larger effective Rabi frequency $\Omega_{\rm eff}$.

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