Simplified measurement of the Bell parameter within quantum mechanics

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We point out that, if one accepts the validity of quantum mechanics, the Bell parameter for the polarization state of two photons can be measured in a simpler way than by the standard procedure [Clauser, Horne, Shimony, and Holt, Phys. Rev. Lett. **23**, 880 (1969)]. The proposed method requires only two measurements with parallel linear-polarizer settings for Alice and Bob at $0°$ and $45°$, and yields a significantly smaller statistical error for a large Bell parameter.

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Measurements of the Bell parameter S for a pair of two-level systems, and verification of Bell's inequality $|S| \leq 2$, were originally designed to experimentally distinguish local hidden-variable theories from quantum mechanics [1, 2]. Many experiments performed since have shown that it is possible to violate Bell's inequality, ruling out local hidden-variable theories, and confirming quantum mechanics [3, 4, 5, 6, 7, 8]. Nevertheless, numerous Bell measurements, with photons [9], ions [8], or combinations of photons and particles [10], continue to be performed. With the validity of quantum mechanics already established (except for the *simultaneous* [11, 12] closing of the locality [7] and detection [8] loopholes), most such measurements serve primarily to confirm the non-classical character of two-particle states. In particular, in quantum cryptography, measurements of the Bell parameter are used to test if the secret key has been compromised [13].

The standard procedure to verify Bell's inequality for photon polarization states, introduced by Clauser, Horne, Shimony and Holt (CHSH) [2], is as follows. The recipients of the two photons, Alice and Bob, measure coincidences for four combinations of linear-polarizer angles: Alice measures at angles $\alpha_1 = 0$ and $\alpha_2 = \pi/4$, and Bob at $\beta_1 = \pi/8$ and $\beta_2 = 3\pi/8$ (see Fig. 1). The Bell parameter is then calculated as

$$S_\pm = \pm[E(\alpha_1, \beta_1) - E(\alpha_1, \beta_2)] + E(\alpha_2, \beta_1) + E(\alpha_2, \beta_2), \quad (1)$$

where

$$E(\alpha_i, \beta_j) = p_+(\alpha_i, \beta_j) - p_-(\alpha_i, \beta_j) \quad (2)$$

is the correlation coefficient of the measurement $\{\alpha_i, \beta_j\}$. Here $p_+(\alpha_i, \beta_j)$ denotes the fraction of events where the polarization measurements by Alice at angle α_i and by Bob at β_j are positively correlated (both photons pass through their respective polarizers, or both are rejected) and $p_-(\alpha_i, \beta_j)$ denotes the fraction of events where the photons are anticorrelated (one passes the polarizer, and the other is rejected). If the photons are perfectly correlated, then $E(\alpha_i, \beta_j) = +1$; for perfectly anticorrelated photons, we have $E(\alpha_i, \beta_j) = -1$. States with $|S| > 2$
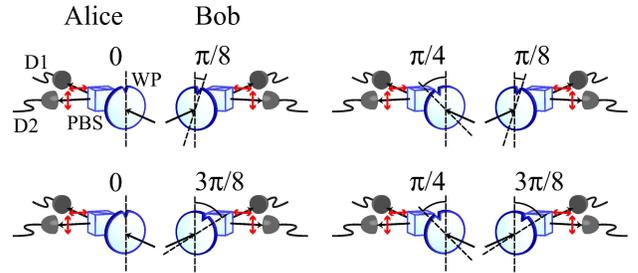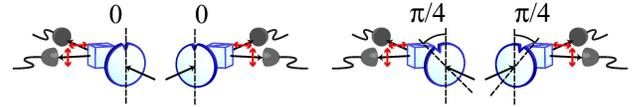

FIG. 1: (Color online) (a) Bell parameter measurement for photon polarizations according to Clauser, Horne, Shimony, and Holt (CHSH), and (b) the proposed alternative scheme. Alice and Bob each receive a photon, and measure its linear polarization. Each detection path consists of a half-wave plate (WP), a polarizing beam splitter cube (PBS), and two detectors (D1 and D2) for the two orthogonal polarizations. The labeled angles are the analysis angles, which are twice the wave plate settings. (a) In the CHSH scheme, Alice and Bob perform measurements with four combinations of angles ($\alpha_1 = 0$ and $\alpha_2 = \pi/4$ for Alice, $\beta_1 = \pi/8$ and $\beta_2 = 3\pi/8$ for Bob), to determine four correlation coefficients $E(\alpha_i, \beta_j)$. They then calculate the Bell parameter as $S_\pm = \pm[E(0, \pi/8) - E(0, 3\pi/8)] + E(\pi/4, \pi/8) + E(\pi/4, 3\pi/8)$. (b) Assuming the validity of quantum mechanics, Alice and Bob can simply use two identical angles, $\alpha_1' = \beta_1' = 0$ and $\alpha_2' = \beta_2' = \pi/4$, and determine the Bell parameter as $S_\pm' = \sqrt{2}[\pm E(0,0) + E(\pi/4, \pi/4)]$.

violate the notion of local realism [1], while the maximum magnitude of the Bell parameter S that quantum mechanics allows is $|S| = 2\sqrt{2}$. Experiments with photons [4, 5, 6], and later with ions [8, 10], have shown that systems yielding $|S| > 2$ can indeed be prepared, with measured values as large as $|S| = 2\sqrt{2} - 0.003(19)$ [14].

The CHSH procedure, using Eq. (1), allows one to ex-

perimentally rule out local hidden-variable theories [1, 2]. If one merely attempts to characterize [15, 16] a two-qubit state within quantum mechanics by the Bell parameter, however, the question arises whether it is *necessary* that Alice and Bob perform four measurements, and along different polarization axes. Assuming the validity of quantum mechanics, we show that it is in fact sufficient for Alice and Bob to measure in the same polarization basis, for two different bases (e.g., $\alpha_1' = \beta_1' = 0$ and $\alpha_2' = \beta_2' = \pi/4$), to determine the Bell parameter $S_\pm' = S_\pm$ more simply as

$$S_\pm' = \sqrt{2}\left[\pm E(\alpha_1', \beta_1') + E(\alpha_2', \beta_2')\right].\tag{3}$$

As discussed below, such a prescription arises naturally from the transformation properties of Bell states under rotations. The proposed method discriminates directly between all four Bell states via the signs in Eq. (3), and requires only two combinations of polarizer settings whereas the CHSH scheme requires four. Finally, our method provides significantly smaller statistical errors for states that violate Bell's inequality.

We should note that, while Eq. (3) represents a simpler measurement of the Bell parameter within quantum mechanics, $S_\pm'$ in this particular form does not rule out local hidden-variable theories. In other words, $S_\pm' = S_\pm$ holds within quantum mechanics, but not generally in local hidden-variable theories. There is currently an interesting dispute [17, 18] as to whether Bell's theorem in a broader sense is valid, or whether non-commuting local realistic models [17] can invalidate the conclusions drawn from standard Bell tests [1, 2, 3, 4, 5, 6, 7, 8] about the non-local or non-realistic character of quantum mechanics. If the latter is the case, or if one merely wishes to characterize two-photon states, it is preferable to use the quantum-mechanically equivalent, simplified form of the Bell parameter, Eq. (3), rather than the CHSH prescription, Eq. (1).

The states that yield the maximum violation of Bell's inequality $|S_\pm| = 2\sqrt{2} \nleq 2$, are the Bell states $\left|\Phi_\pm^{HV}\right\rangle$, $\left|\Psi_\pm^{HV}\right\rangle$, obtained by applying the Bell-state creation operators

$$\left(\hat{\Phi}_\pm^{HV}\right)^\dagger = \frac{\hat{h}_A^\dagger \hat{h}_B^\dagger \pm \hat{v}_A^\dagger \hat{v}_B^\dagger}{\sqrt{2}}$$
$$\left(\hat{\Psi}_\pm^{HV}\right)^\dagger = \frac{\hat{h}_A^\dagger \hat{v}_B^\dagger \pm \hat{v}_A^\dagger \hat{h}_B^\dagger}{\sqrt{2}},\tag{4}$$

to the photon vacuum $|0\rangle$,

$$\left|\Phi_\pm^{HV}\right\rangle \equiv \left(\hat{\Phi}_\pm^{HV}\right)^\dagger |0\rangle = \frac{|hh\rangle \pm |vv\rangle}{\sqrt{2}}$$
$$\left|\Psi_\pm^{HV}\right\rangle \equiv \left(\hat{\Psi}_\pm^{HV}\right)^\dagger |0\rangle = \frac{|hv\rangle \pm |vh\rangle}{\sqrt{2}}.\tag{5}$$

Here $|ab\rangle \equiv |a\rangle_A |b\rangle_B$ denotes the polarizations a, b of the photons received by Alice (A) and Bob (B), respectively. We have defined $|h\rangle_A = \hat{h}_A^\dagger |0\rangle$, with $\hat{h}_A^\dagger$ being the

creation operator for one horizontally polarized photon in Alice's mode, with similar definitions for $|v\rangle_A, |h\rangle_B$ and $|v\rangle_B$. The HV superscript indicates that these Bell states are defined in the horizontal/vertical (HV) basis.

We now consider a linear polarization basis ST (for both Alice and Bob) at $45°$ relative to the HV basis, defined by the photon creation operators

$$\hat{s}_\mathcal{K}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{h}_\mathcal{K}^\dagger + \hat{v}_\mathcal{K}^\dagger\right)$$
$$\hat{t}_\mathcal{K}^\dagger = \frac{1}{\sqrt{2}}\left(-\hat{h}_\mathcal{K}^\dagger + \hat{v}_\mathcal{K}^\dagger\right),\tag{6}$$

where $\mathcal{K} = A, B$. It is then easy to see the correspondence of the Bell states between the two bases:

$$\left|\Phi_+^{ST}\right\rangle \equiv \frac{|ss\rangle + |tt\rangle}{\sqrt{2}} = \frac{|hh\rangle + |vv\rangle}{\sqrt{2}} = \left|\Phi_+^{HV}\right\rangle$$

$$\left|\Phi_-^{ST}\right\rangle \equiv \frac{|ss\rangle - |tt\rangle}{\sqrt{2}} = \frac{|hv\rangle + |vh\rangle}{\sqrt{2}} = \left|\Psi_+^{HV}\right\rangle$$

$$\left|\Psi_+^{ST}\right\rangle \equiv \frac{|st\rangle + |ts\rangle}{\sqrt{2}} = \frac{-|hh\rangle + |vv\rangle}{\sqrt{2}} = -\left|\Phi_-^{HV}\right\rangle$$

$$\left|\Psi_-^{ST}\right\rangle \equiv \frac{|st\rangle - |ts\rangle}{\sqrt{2}} = \frac{|hv\rangle - |vh\rangle}{\sqrt{2}} = \left|\Psi_-^{HV}\right\rangle.\tag{7}$$

The above relations show that, e.g., the Bell state $\left|\Phi_+^{HV}\right\rangle$ has maximum positive correlation in the HV basis (Alice and Bob always measure the same polarization), as well as in the ST basis, while the state $\left|\Phi_-^{HV}\right\rangle$ displays maximum positive correlation in the HV basis, and maximum anticorrelation in the ST basis. In contrast, a mixed state with equal probability of $|hh\rangle$ and $|vv\rangle$ pairs shows no correlations in the ST basis. Therefore one may suspect that the Bell states can be identified by the combination of correlation measurements in the HV and ST bases, and that it may not be necessary for Alice and Bob to measure at nonzero relative polarizer angles.

To demonstrate the equivalence of Eqs. (1) and (3) for measuring the Bell parameter, we show that both expressions represent the expectation value of the same Bell operator [19]. For an arbitrary (pure or mixed) photon-pair input state as described by a density matrix ρ, the correlation coefficient $E(\alpha_i, \beta_j)$ can be written as the expectation value $E(\alpha, \beta) = \text{Tr}\left(\rho\hat{\mathcal{E}}_{\alpha\beta}\right)$ of the operator

$$\hat{\mathcal{E}}_{\alpha\beta} = \hat{\alpha}^\dagger\hat{\alpha}\hat{\beta}^\dagger\hat{\beta} + \hat{\alpha}_\perp^\dagger\hat{\alpha}_\perp\hat{\beta}_\perp^\dagger\hat{\beta}_\perp - \hat{\alpha}_\perp^\dagger\hat{\alpha}_\perp\hat{\beta}^\dagger\hat{\beta} - \hat{\alpha}^\dagger\hat{\alpha}\hat{\beta}_\perp^\dagger\hat{\beta}_\perp$$
$$= \left[\left(\hat{h}_A^\dagger\hat{h}_A - \hat{v}_A^\dagger\hat{v}_A\right)\cos 2\alpha + \left(\hat{h}_A^\dagger\hat{v}_A + \hat{v}_A^\dagger\hat{h}_A\right)\sin 2\alpha\right]$$
$$\times \left[\left(\hat{h}_B^\dagger\hat{h}_B - \hat{v}_B^\dagger\hat{v}_B\right)\cos 2\beta + \left(\hat{h}_B^\dagger\hat{v}_B + \hat{v}_B^\dagger\hat{h}_B\right)\sin 2\beta\right].\tag{8}$$

Here $\hat{\alpha} = \hat{h}_A \cos\alpha + \hat{v}_A \sin\alpha$ and $\hat{\beta} = \hat{h}_B \cos\beta + \hat{v}_B \sin\beta$ are the annihilation operators for a photon polarized along angles α in Alice's and β in Bob's mode, respectively, while $\alpha_\perp = \alpha + \pi/2$ and $\beta_\perp = \beta + \pi/2$ denote the corresponding orthogonal polarizations. Eq. (8) follows directly from the definition of the correlation coefficient

[Eq. (2)], since $\hat{a}_A^\dagger \hat{a}_A \hat{b}_B^\dagger \hat{b}_B$ measures the fraction of photon pairs with polarizations a, b, observed by Alice and Bob, respectively. The Bell parameter is the expectation value $S_\pm = \mathrm{Tr}\left(\rho \hat{S}_\pm\right)$ of the operator

$$\hat{S}_\pm = \pm\left(\hat{\mathcal{E}}_{0,\frac{\pi}{8}} - \hat{\mathcal{E}}_{0,\frac{3\pi}{8}}\right) + \hat{\mathcal{E}}_{\frac{\pi}{4},\frac{\pi}{8}} + \hat{\mathcal{E}}_{\frac{\pi}{4},\frac{3\pi}{8}}, \qquad (9)$$

which, using Eq. (8) and some trigonometric relations, can be rewritten as

$$\begin{aligned}
\hat{S}_\pm &= \pm\sqrt{2}\left(\hat{h}_A^\dagger \hat{h}_B^\dagger \pm \hat{v}_A^\dagger \hat{v}_B^\dagger\right)\left(\hat{h}_A \hat{h}_B \pm \hat{v}_A \hat{v}_B\right) \\
&\quad \mp\sqrt{2}\left(\hat{h}_A^\dagger \hat{v}_B^\dagger \mp \hat{v}_A^\dagger \hat{h}_B^\dagger\right)\left(\hat{h}_A \hat{v}_B \mp \hat{v}_A \hat{h}_B\right) \\
&= \pm 2\sqrt{2}\left[\left(\hat{\Phi}_\pm^{HV}\right)^\dagger \hat{\Phi}_\pm^{HV} - \left(\hat{\Psi}_\mp^{HV}\right)^\dagger \hat{\Psi}_\mp^{HV}\right].(10)
\end{aligned}$$

Since $\hat{n}_{\Phi_\pm} = \hat{\Phi}_\pm^\dagger \hat{\Phi}_\pm$ and $\hat{n}_{\Psi_\pm} = \hat{\Psi}_\pm^\dagger \hat{\Psi}_\pm$ give the probability of observing the corresponding Bell state when evaluated in any photon-pair state, Eq. (10) shows that the Bell parameter measures the difference in occupation of pairs of Bell states, namely $S_+ = 2\sqrt{2}\left(n_{\Phi_+} - n_{\Psi_-}\right)$ or $S_- = 2\sqrt{2}\left(n_{\Psi_+} - n_{\Phi_-}\right)$.

On the other hand, the simplified Bell operator

$$\hat{S}_\pm' = \sqrt{2}\left(\pm\hat{\mathcal{E}}_{0,0} + \hat{\mathcal{E}}_{\frac{\pi}{4},\frac{\pi}{4}}\right), \qquad (11)$$

whose expectation value is given by Eq. (3), can also be expressed with the help of Eq. (8) as

$$\hat{S}_\pm' = \pm 2\sqrt{2}\left[\left(\hat{\Phi}_\pm^{HV}\right)^\dagger \hat{\Phi}_\pm^{HV} - \left(\hat{\Psi}_\mp^{HV}\right)^\dagger \hat{\Psi}_\mp^{HV}\right]. \qquad (12)$$

This establishes for the operators underlying Eqs. (1), (3) the identity $\hat{S}_\pm' = \hat{S}_\pm$, which implies the same relation for all expectation values. To derive this identity, we have used the laws of quantum mechanics: We made use of the linear transformation for fields or operators, Eq. (6), that gives rise to Malus' law, as well as the operator expectation values to connect Eqs. (9), (11) to the measured quantities. Consequently, if one accepts hidden-variable theories as ruled out by experiments [4, 5, 6], and quantum mechanics to be valid, one can measure the Bell parameter using the simplified expression Eq. (3), rather than the CHSH formula Eq. (1).

In order to elucidate the connection between the Bell operator $\hat{S}_\pm' = \hat{S}_\pm$ and the nature of the quantum correlations, we introduce the coincidence operator

$$\hat{f}_{ab} = \hat{a}_A^\dagger \hat{a}_A \hat{b}_B^\dagger \hat{b}_B. \qquad (13)$$

This operator measures the fraction $f_{ab} = \langle \hat{f}_{ab} \rangle = \mathrm{Tr}\left(\rho \hat{f}_{ab}\right)$ of photon pairs with polarizations a, b observed by Alice and Bob, respectively (e.g., f_{hv} is the fraction of events where Alice observes a horizontally polarized photon and Bob observes a vertically polarized photon, when they both perform measurements in the HV basis). Using Eqs. (4), (12) and rearranging terms, the Bell

TABLE I: Relation between Bell states and Bell parameter $S_\pm' = S_\pm$.

Bell states	$S_+' = S_+$	$S_-' = S_-$
$\left\|\Phi_+^{HV}\right\rangle = \left\|\Phi_+^{ST}\right\rangle$	$2\sqrt{2}$	0
$\left\|\Psi_+^{HV}\right\rangle = \left\|\Phi_-^{ST}\right\rangle$	0	$2\sqrt{2}$
$\left\|\Phi_-^{HV}\right\rangle = -\left\|\Psi_+^{ST}\right\rangle$	0	$-2\sqrt{2}$
$\left\|\Psi_-^{HV}\right\rangle = \left\|\Psi_-^{ST}\right\rangle$	$-2\sqrt{2}$	0

parameter can be expressed in terms of the coincidences as

$$\frac{S_\pm'}{\sqrt{2}} = \pm\left(f_{hh} + f_{vv} - f_{hv} - f_{vh}\right) + \left(f_{ss} + f_{tt} - f_{st} - f_{ts}\right). \qquad (14)$$

Coincidences in one basis, such as f_{hh} or f_{hv}, can arise from either classical or quantum correlations. Only quantum correlations can, however, appear simultaneously in two different bases such as HV and ST. In fact, the combination of correlations in the two bases distinguishes completely the underlying Bell state. For instance, Eq. (7) shows that the state $\left|\Psi_+^{HV}\right\rangle = \left|\Phi_-^{ST}\right\rangle$ has maximum hv, vh, ss, and tt coincidences, but no hh, vv, st, or ts coincidences. Correspondingly, the quantity S_-' will be the largest in magnitude, and equal to $2\sqrt{2}$. The relation between the Bell parameter $S_\pm' = S_\pm$ and the Bell states in the two bases, Eq. (7), is summarized in Table I.

The proposed scheme proves to have smaller errors, both systematic and statistical, compared to the CHSH scheme. We first note that the systematic error, mainly due to the inaccuracy of the setting of the wave plates, should be lower in most cases for the simplified scheme, as it involves only two combinations of angles, $\alpha_1' = \beta_1' = 0$ and $\alpha_2' = \beta_2' = \pi/4$. To evaluate the statistical error, we assume that the Bell parameter is close to its maximum value $|S_\pm| = |S_\pm'| \sim 2\sqrt{2}$. We then find that, for a total of N detected photon pairs, the variance $(\Delta S)^2$ of the CHSH form of the Bell parameter $S_\pm$ is given by

$$(\Delta S)^2 = \frac{4}{N} + \mathcal{O}\left(\frac{1}{N^2}\right), \qquad (15)$$

while, for the simplified form $S_\pm'$, we obtain the expression

$$(\Delta S_\pm')^2 = \frac{16}{N}\left(1 - \frac{|S_\pm'|}{2\sqrt{2}}\right) + \mathcal{O}\left(\frac{1}{N^2}\right). \qquad (16)$$

This shows that our proposed scheme provides a significantly smaller statistical error for a Bell parameter exceeding $\frac{3}{4} \times 2\sqrt{2}$, or $|S_\pm'| > 2.12$.

Another possible way to measure the Bell parameter is to observe the visibility of the interference fringes of coincidence rates vs. angle β for two fixed settings $\alpha_1 = 0$ and $\alpha_2 = \pi/4$, and to calculate the Bell parameter as the average visibility of the two fringes [9, 20]. This is only possible, however, if the state is known to be rotationally symmetric, and if the correlation coefficient

$E(\alpha_i, \beta)$ varies sinusoidally with β. In contrast, our method Eq. (3) applies generally, and is not restricted to the rotationally symmetric state.

In conclusion, we have pointed out an alternative scheme, within quantum mechanics, for measuring the Bell parameter. The scheme requires only two different combinations of polarizer settings for Alice and Bob and gives smaller measurement errors compared to the standard CHSH scheme. We have shown how this method arises naturally from the transformation of Bell states under rotations. This simplifies the characterization of non-classical two-photon states, in particular for photons travelling along the same path [21].

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